Aristotle on Continuity[1]

My topic in this paper is Aristotle's understanding of mathematical continuity. While the idea that continua are composed of infinite points is the present day orthodoxy, the Aristotelian understanding of continua as non-punctiform and infinitely divisible was the reigning theory for much of the history of western mathematics, and there is renewed interest in it from current mathematicians and philosophers of mathematics.[2] In this paper I will make use of Aristotle's discussion of unity in Book V of his *Metaphysics* to argue that the notion of a *per se* unity, something which is a unity through itself or *kath autō*, is at the heart of Aristotle's understanding of continuity and continua; continua are *per se* unities. I will use this insight to offer an interpretation of Aristotle's discussion of continuity in *Physics* V & VI, and to solve one of the puzzles which arises from this discussion.

There are several puzzles to be solved in the *Physics* discussion of continuity; I am concerned here with whether it is successful in distinguishing continuity from contiguity among pure geometrical items such as successive lines, figures, or solids. The difficulty is as follows. Aristotle distinguishes the way in which successive magnitudes are related so as to be continuous from the way in which two contiguous magnitudes are related. In his analysis, contiguous magnitudes touch at some extremity while the boundaries at which continuous magnitudes touch actually become one (227a10-13). The latter claim is his operative definition of continuity.

[1] I owe thanks to Fred Miller, Sarah Broadie, Marco Panza, Neil Tennant, and Stewart Shapiro for comments upon earlier drafts of this paper.

[2] E.g Stewart Shapiro, Geoffrey Hellman, and Oystein Linnebo.

Aristotle goes on to claim that magnitudes can be contiguous without thereby being continuous. (227a21-23) This seems straightforwardly true when applied to bodies, but among pure geometrical magnitudes it seems *prima facie* as though some contiguous magnitudes are continuous with each other just in virtue of their contiguity; when their boundaries touch they coincide and so seem to become one. Employing my interpretation of continua as *per se* unities, and drawing insight from Lear's 1982 discussion of Aristotelian mathematical ontology, I will argue that Aristotle's distinction between continuity and contiguity is sufficient for abstract geometrical items as well as for bodies.

A clarification should be made here. In Aristotle's treatment contiguity is a relation which holds between magnitudes, e.g., between two or more figures or bodies. Whether continuity is also a relation is at first unclear; Aristotle uses the term "continuous" in two ways. Sometimes, as for example in the first lines of *Physics* VI.1, he uses the term "continuous" in a relational sense. At 231a21-23 Aristotle says that two magnitudes stand in a relation of continuity when the extremities at which they touch are one. In the next line, however, Aristotle indicates that it follows that "nothing that is continuous can be composed of indivisibles: e.g. a line cannot be composed of points." [3] (231a24-25) Here, evidently, Aristotle uses the term "continuous" to refer to a property which belongs to a magnitude in isolation. The difficulty is that both when he refers to the relation and when he discusses the non-relational property Aristotle appeals to the same definition of things being continuous, the one given at 227a10-13.

[3] Unless otherwise indicated, in what follows all translations will be from the R.P. Hardie and R.K. Gaye translations of the *Physics* and *Metaphysics* in *The Complete Works of Aristotle* Vol I&II. Ed. Jonathan Barnes, Princeton, Princeton University Press (1984)

Nevertheless, however confusingly Aristotle seems to conflate these two, a consistent position can be pieced together. In the first section of this paper I will argue that a comparison of the *Physics* passage with the discussion of unity in Aristotle's *Metaphysics*, especially *Metaphysics* V, shows that non-relational or holistic continuity is a kind of *per se* unity. In the following section I will turn to [relational] continuity and Aristotle's distinction between it and contiguity. I argue that understanding what happens when the boundaries of contiguous magnitudes become one shows that *per se* unity belongs to wholes which are formed when the boundaries of their parts become one. In other words, while contiguous magnitudes are such that they are in succession and their boundaries are together, something is [non-relational] continuous when the boundaries of its successive parts not only touch but actually become one, *i.e.*, when its parts are [relational] continuous with each other. This, I argue, is what Aristotle has in mind when he claims at 227a15 that "This definition [the definition of continuity] makes it plain that continuity belongs to things that naturally in virtue of their mutual contact form a unity."

I

In *Metaphysics* V Aristotle distinguishes several ways in which something can be one. The first two ways are relevant here:

> We call one (1) that which is one by accident, (2) that which is one by its own nature. … Of things that are called one in virtue of their own nature some (*a*) are so called because they are continuous, e.g., a bundle is made one by a band, and pieces of wood are made one by glue… Those things are continuous by their own

> nature which are one not merely by contact; for if you put pieces of wood touching one another, you will not say these are one piece of wood or one body or one *continuum* of any other sort. Things, then, that are continuous in any way are called one." (1016a1, 5, 7-9)[4]

What is translated here as something being one "by its own nature" rather than accidentally is the Greek phrase *kath' auto,* of which the Latin *per se* is a literal (and more familiar) translation. The claim is that things which are continuous are, in virtue of being continuous and thus in virtue of themselves, or *per se*, unified wholes. Continua are *per se* unities.

The difference between *per se* unity and accidental unity is best illustrated by examples. Examples of accidental unities are such as piles of sand, stacks of wood, or handfuls of straw. Intuitively these are not really one; they are multitudes of grains of sand or pieces of straw, etc. One may consider these groups of things as though they form a whole when the parts are gathered together and put side by side. In this way I can refer to and think of something as "this stack of wood," and I can distinguish it from some other stack of wood. But what I refer to when I speak this way is still a stack or a heap – a bunch of independent things which happen to be put together.

Continuing the example, what would it take to make the stack of wood a whole? Binding or gluing the sticks together. Then instead of a heap of individual sticks of wood, there would be a single thing whose unity is not just a matter of proximity. An example Aristotle uses in his Book V discussion is that of a shoe; we do not usually think of the parts of a shoe as forming a continuous whole unless they are arranged appropriately and bound together. Put together in just

[4] Trans.W.D. Ross, in Vol. II of *The Complete Works of Aristotle.*

any way whatever, what is produced is not a whole but a sole, laces, and upper in a pile. (1016b15) Later in his *Metaphysics* Aristotle offers the example of a syllable. A syllable, he says, is more than its component parts. The syllable "ba" has the component parts of b and a. But it is not merely b+a, for then "ba" and "ab" would be the same syllable; "the syllable, then is something—not only its elements (the vowel and the consonant) but also something else." (1041b11) In these cases, contact or grouping of the parts is not sufficient to produce a continuous whole. Something more is required to bind them or unify them. For Aristotle, this something more is not merely another element in the compound, but something which enables the whole to hold itself together as one. In the case of the syllable it is the order of the parts; in the case of the stack of wood it is the glue; for the shoe it is the function and the arrangement which is required to fulfill this function.

The idea is that things which are continuous in some sense hold themselves together, while contiguous things are not similarly responsible for their being together. There is an interesting way in which this comparison is exhibited by the etymology of the Greek words. The Greek word for contiguous is *echomenon*; the word for continuous is *suneches*. Both of these come from the same root, the verb *echō*, which means, roughly, to have or hold in hand. *Echomenon* is the passive participle. *Suneches*, on the other hand, is the active participle from *echo* with the prefix *sun* ("with" or "together"). A *suneches* or continuous thing is something which holds together; here, it holds itself together. Cornford notes the same thing in a footnote to the Greek text in the Loeb edition of the *Physics*. Comparing the Greek etymology of *suneches* to the Latin etymology of "continuous," he says the following:

> *Cf. continent* = continuous land unparted by sea, a 'continent' person, one who can 'hold himself together.' In Greek and Latin the etymological implication of the phrase is more general and obvious than in English." [5]

This comes through in the *Physics* V&VI discussion. Having claimed that something is not a continuum if the boundaries of the parts are two and not one, Aristotle goes on to say that "[T]his definition makes it plain that continuity belongs to things that naturally in virtue of their mutual contact form a unity. And in whatever way that which holds them together is one, so too will the whole be one, e.g. by a rivet or glue or contact or organic union." (227a15) The translation just quoted may be misleading; the idea that the parts touching each other ("mutual contact") is sufficient to unify them as a continuum seems to elide just the distinction between the continuous and the contiguous for which Aristotle is arguing. But it need not. When at 227a9 Aristotle defines a contiguous magnitude as one which "touches" another, the word which "touches" translates is *haptētai* from *haptō* ("to touch"). The word which Hardie & Gaye translate as "mutual contact" in 227a15 is *sunapsis*. It comes from *haptō* with the prefix *sun*; while literally a *sunapsis* is a touching-together, or a mutual contact, the verb *sunaptō*, from which the noun is derived, usually means something a bit stronger. The Liddell and Scott lexicon offers as its first meaning "to join together," or "to bind." The idea, then, is not that it is merely by the parts touching or being contiguous that they form a continuum, but that it is by the joining or binding of what is already in a sense together that a continuum arises. The use of the stronger word, *sunapsis*, helps to capture the thought that there is a continuum only when the

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[5] Aristotle. *The Physics: Books V-VIII.* Trans. Philip H. Wicksteed and Francis M. Cornford. Cambridge, Harvard University Press (1934): pp. 38, footnote *b*

boundaries of the parts do not merely touch but become one in such a way that the magnitudes together form a new unity.

The consideration of continua as *per se* unities prepares the way for the idea of potential parts. The slogan for potential parts is straightforward enough: "things that are thus actually two are never actually one." (1039a3) But something can be actually one and potentially two; this is the way in which continua have parts, that is, by having potential parts.

An argument for continua having parts only potentially can be developed by analogy with the argument from 1039a3-10 that if the parts of substances are also substances, then those parts can exist only potentially. Heaps or piles of things are actually, in virtue of themselves, multitudes; it is only accidentally that they are one. But continua, especially things which are *per se* continuous (1016a4), are actually, in virtue of themselves, unities. Furthermore, the parts of continua are also [non-relational] continuous in themselves, since lines are non-punctiform and infinitely divisible. Thus the parts of continua are also *per se* unities. If the parts are actual then continua are both actually one and actually many. Thus continua must have parts only potentially. [6]

II

I will turn now to the way in which Aristotle draws the distinction between contiguous and [relational] continuous magnitudes in *Physics* V & VI in order to link together relational and non-relational continuity. Aristotle begins his exposition of continuity at 227a9 explaining that "A thing that is in succession and touches [*haptētai*] is contiguous [*echōmenon*]." To be in

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[6] *Cf.* also the argument that the parts of living things and souls can only exist potentially, since living things and souls are "one and continuous by nature." (1040b5-15).

succession in the relevant respect, Aristotle says, is to be after something else such that nothing else of the same kind is between the thing and what precedes it. (226b35) For something to touch [*haptō*] another is for its boundary to be together in place with the boundary of the thing it touches. (226b20)[7]

Aristotle's complete definition of the continuous is as follows:

> The continuous [*suneches*] is a subdivision of the contiguous [*echōmenon*]: things are called continuous when the touching [*haptontai*] limits of each become one and the same and are (as the word implies) contained in each other [*sunechētai*]: continuity is impossible if these extremities are two. (227a10-13)

Magnitudes are contiguous when their boundaries are together in place, or touch; they are continuous when they not only touch but their boundaries where they touch become one.[8]

Can the boundaries of magnitudes can be together in place without becoming one – is Aristotle's treatment of contiguity consistent with his claim that "the extremities of things may be together without necessarily being one" (227a22)? The extremities of two bodies, two

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[7] Aristotle clarifies that by "together" with respect to place he means "in the same primary place." (226b21-22) This is fmaously problematic clarification if it is taken strictly. Aristotle defines primary place as the innermost boundary of the containing body (212a20). Strictly speaking, points thus cannot be in place. Aristotle usually seems loathe to claim that they are; e.g., at 212b24, where he says that "There is no necessity … that a point should have a place", following which he argues that only moveable bodies are in place (212b29). Difficulties about Aristotle's account of place are too tangled and written-over to be entered into in this paper, however. Further, if we take him not to be speaking strictly about place in this passage, I think that the sort of thing Aristotle is gesturing at when he says that boundaries are together in place is clear enough for the present project.

[8] In his commentary on the *Physics* Ross (1936, pp. 626) offers two ways of categorizing magnitudes according to the terms in use in *Physics* V & VI. According to both of these categorizations there are two species of the contiguous: what is continuous and what is not continuous.

sensible magnitudes, can be together without being one. Consider two books side by side. Setting aside the fact they their covers may no longer be quite smooth, so that the surfaces which bound them do not quite touch along their length, it seems perfectly natural to think that they are contiguous. They are in succession, one standing next to the other with no other books in between, and their boundaries are touching on one side. Further, the cover of the first book is not the same as the cover of the second; it is because the surfaces are the boundaries of distinct bodies that they are not one. The books are contiguous, but are not continuous with each other.

The question is whether the boundaries of any pure geometrical items, such as straight lines, figures, or solids, can touch or be together in place without also becoming one. In the case of lines and figures, it seems that for boundaries to touch is for them to coincide. But when geometrical items coincide it is not clear how their numerical distinctness can be preserved.

That the boundaries of geometrical items coincide when they are together in place is clear. According to Aristotle, points are the extremities or boundaries of line segments. (*Categories* 5a2[9]; cf. Euclid's *Elements*[10], Book I def. 3) They are also indivisible and have no parts. (*Physics* 231b1; cf. Euclid's *Elements*, Book I def. 1) Since they have no parts, points can only touch as whole to whole rather than as part to part or part to whole. Thus whenever they touch, as Aristotle seems to allow in *Metaphysics* 1069a12, they coincide. Furthermore, points cannot be in successive places, since between any two points there is a line. (Euclid, Book I, postulate 1) Since the boundaries of lines are points, then whenever the limits of two successive lines are together and touch, they also coincide. Analogous arguments can be made, *mutatis mutandis*, for contiguous figures and solids.

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[9] In *The Complete Works of Aristotle, vol. I.*

[10] Euclid. *Elements*, trans. Sir Thomas L. Heath, in *Great Books*, Vol. 11, ed. Robert Maynard Hutchins. Chicago, Encyclopaedia Britannica, Inc. (1952)

If boundaries coinciding is sufficient for their becoming one it seems that geometrical things cannot be contiguous without also being continuous with each other. This would contradict Aristotle's claim that "if there is contact, that alone does not imply continuity; for the extremities of things may be together without necessarily becoming one." (227a22-23)

There is textual evidence that Aristotle does not think that the coincidence of boundaries is, or is always, sufficient for their becoming one. In his discussion of place in *Physics* IV Aristotle allows that the innermost boundary of the containing body and the boundary of the contained body coincide (211b10-14), but argues that they are nevertheless not the same (211b13) because they are the boundaries of different bodies. The numerical distinctness of the boundaries follows from the numerical distinctness of the bodies they bound.

We should be able to apply the same sort of consideration to the case of pure geometrical items such as straight lines. According to Aristotle boundaries are accidental beings which depend for their existence upon the things which they bound. Thus points, the boundaries of lines, are accidental beings individuated by the lines they bound. Thus to apply Aristotle's argument from book IV to the boundaries of abstract geometrical items we must find an appropriate way to distinguish those geometrical items. In the case of sensible magnitudes the bodies can be distinguished by their physical characteristics. *E.g.*, one is made of bronze and the other consists of water, or one is wooden and the other is a nail which has been nailed into the first. The difficulty for the discussion of geometry is that these characteristics are abstracted from or, to adapt Lear's expression[11], filtered out, by considering the magnitudes only *qua* geometrical. It is not clear what else could guarantee their distinctness. If we cannot find an

[11] Lear, Jonathan. "Aristotle's Philosophy of Mathematics." The Philosophical Review, Vol. 91, No. 2 (Apr., 1982), pp. 161-192

independent way of distinguishing the contiguous magnitudes, then we will not be able to use their numerical distinctness to get the numerical distinctness of their coincident boundaries.

If Aristotle's view fails with respect to pure magnitudes, one might wonder whether it is intended to apply to pure mathematical items. After all, the *Physics* is not a treatise in mathematics; it is aimed at the understanding of natural, material things. If the *Physics* treatment of continuity is not concerned with abstract geometrical items then the fact that his assessment of continuity fails to apply to them in a satisfactory way is not a problem.

However, the properties which Aristotle ascribes to continuous bodies or things in the *Physics* V passage are exactly the ones he makes use of to argue in the very next book of the *Physics* that lines, in particular, cannot be composed of points, nor can any continuous thing be composed of indivisibles. He clearly has in mind here abstract geometrical items. Consequently, Aristotle's distinction between the contiguous and the continuous cannot be saved by assuming it was meant to apply only to bodies but not to geometrical items *qua* geometrical. At least *prima facie* Aristotle's discussion takes as its subject both sensible and pure magnitudes. Something more must be said.

I think a solution can be found by reflecting upon Aristotle's mathematical ontology. I follow Jonathan Lear in his 1982 "Aristotle's Philosophy of Mathematics" to suppose that, according to Aristotle, what geometry studies are material, physical things but not *qua* physical. Geometry studies sensible magnitudes, but it attends to them only insofar as they have magnitude or are geometrical, not insofar as they are material, sensible, etc. Within the study of geometry one applies what Lear called a "predicate filter" (Lear, pg. 168 -169) to attend only to those properties which follow from the object being extended and shaped:

> Thus, for Aristotle, one can say truly that separable objects and mathematical objects exist, but all this statement amounts to – when properly analyzed – is that mathematical properties *are truly instantiated in physical objects*[12] and, by applying a predicate filter, we can consider these objects as solely instantiating the appropriate properties. (Lear, pg. 170)[13]

Perhaps all that is needed for the present difficulty is to note that the bodies which we are examining mathematically are themselves distinct, and their mathematical properties can be differentiated by the bodies they belong to.[14] If we begin our geometrizing with, for example, two figures X and Y, which are numerically two because they are instantiated in distinct bodies, nothing about our geometrical analysis of these should change the fact that they are two.

Something like this might help to support a suggestion made by Marco Panza in a so far unpublished note on Aristotelian continuity.[15] Using the example of two straight lines which share a common point as a boundary, he argues that one can functionally distinguish the endpoint of one from the coinciding endpoint of the other. When two lines, A and B, seem to share a point as a common boundary, they share the same point in the sense that the same location limits both lines. But points are individuated by what they do, that is, by which line they limit. At the end of A is a point, c, which delimits A; at the end of B is a point, d, which delimits B. These cannot be the same point, even if they coincide and so are in some sense one, for they serve different

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[12] Emphasis mine

[13] Cf. Corkum, Phil. "Aristotle on Mathematical Truth." *British Journal for the History of Philosophy*, vol. 20: pp. 1057-1076.

[14] Admittedly this touches upon the discussion of Aristotelian non-substantial particulars, for which there is a vast literature. Here I presuppose that non-substantial particulars are non-recurrent. For a recent defense of this position, see Wedin (2000).

[15] Panza, Marco. "Aristotle's Continuity." Unpublished. Nov. 2011. Used with permission from the author.

functions, namely, the limiting of different lines. This is like the case of the coincident boundaries of bodies from *Physics* IV.

What happens when the boundaries become one? It cannot be that only one of the lines is bounded; for Aristotle all lines are actually finite. But if the one point bounds neither line then it no longer marks an *actual* division between magnitudes; at best it marks a place where there *could be* a division – the point is only potential. When the boundaries of magnitudes become one in the way required, they disappear as actual boundaries, so that they, and the parts they delimit, are merely potential. Thus when two magnitudes are [relational] continuous, they form (potential) parts of a single continuous whole.

On this reading, then, the same distinction which holds for continuous and contiguous bodies holds also for continuous and contiguous quantities. A continuous whole, whether a body or a purely geometrical item, is an undivided *per se* unity which has parts only potentially. Further, a continuum is such that when successive parts are actualized they bear toward each other the relation of contiguity.

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