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Counter-Intuitionist: A Defense of Classical Quantifier Exchange

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??????????? The intuitionist movement?s roots lie in constructivist mathematics.*[4]*? Accordingly, intuitionists insist on a constructivist conception of proof.*[5]*? Michael Dummett?s theory of verification provides a clear application of these principles to statements: ?to know the meaning of a statement is, on such a view, to be capable of recognising whatever counts as verifying the statement, i.e. as conclusively establishing it as true.? *[6]* ?More recently, Dummett further explicates this position under the term ?justificationism?: ?The justificationist thinks that the notion of truth must be explained in terms of that of justification.? He takes a statement to be true only if there exists something that would justify its assertion, and similarly for the truth of a thought.?*[7]*? He differentiates between mathematical and empirical statements, asserting that mathematical statements require conclusive proof while empirical statements may at times be justified through inductive arguments;*[8]* this paper will focus on empirical statements.

In his discussion of these two types of statements, Dummett notes another crucial difference: empirical statements fall under temporal restrictions that do not necessarily hold for mathematical statements.*[9]* ?I would

like to call attention to two allowances Dummett provides in response to this issue:

(i) I will readily grant that statements about observable states of affairs as occurring at various places and times will always be capable in principle of being known or reasonably judged to be true, though in practice many will not.*[10]*

(ii) The justificationist considers a statement to be true if there exists a direct justification of it, but he allows that we are not in possession of such a direct justification for a great many statements which we accept and can justify only indirectly. The indirect justification does not provide us with the means of obtaining a direct justification, but when it is sound, we take it as guaranteeing that a direct justification exists.? Here ?exists? must be taken as in what Frege called ?the tense of timelessness?, since a direct justification of a statement about the past will usually itself lie in the past.*[11]*

If we deny (ii), then we must also deny (i); if we do not accept indirect justification, then we are left only with what we may actually and in practice be able to judge true. Interestingly, Dummett explains the intuitionist rejection of bivalence in terms of there ?not need[ing] to be anything that would justify either the assertion or denial of a statement; in this case the statement will not be either true or false, although we shall not be able to know that this is so.?*[12]*? While I am unclear how Dummett seeks to reconcile the position he presents against bivalence with (i) and (ii), it is clear that he wishes to allow indirect justification, especially as regards past states of affairs.? ?[The intuitionist] has now abandoned the claim that a statement is true only if we are *now* in a position to find a justification for asserting it true; it is true if there *could have been* such a justification.?*[13]*? But Dummett denies that this leads to full-fledged realism because it involves a counterfactual which relies on a justification of its assertion, and the intuitionist, as noted above, need not accept that all statements are necessarily true or false.

???????????? Dummett provides the following definitions of direct and indirect justification:

A direct justification for a statement about an observable state of affairs can consist only in an observation, actual or possible.? Any argument that it holds good will always be an indirect justification for it.? We must bear in mind that an indirect justification for an empirical statement will not, in general, afford a means of obtaining a direct justification.? A direct justification for a statement that cannot serve as a report of an observation may, and usually will, contain an element of reasoning, but one provided for in the specification of its meaning; an indirect justification for it will consist of an argument not so provided for.*[14]*

I see no problem accepting Dummett?s definitions of these terms.? I take Dummett to allow for discussion of empirical statements without a need for continuous attention and reference to spatio-temporal complications. I single this out for comment because I wish to engage in such discussion of empirical statements without qualifying or restricting my comments and conclusions according to spatio-temporal minutiae.? While he makes a distinction between direct and indirect justification, stating that, strictly speaking, a justificationist must rely on the former for decisions regarding truth, Dummett allows that soundness of argument based on indirect justification goes a long way in establishing that a direct justification exists even if not currently available.? He also accepts a range of methods of proof, including observational data, deductive proofs and processes of reasoning.*[15]*? In light of these allowances, I am confident that justificationists like Dummett, and by extension intuitionists, will accept sound arguments based on indirect proof as establishing the truth of statements.? I will return to this point later in this discussion.

??????????? Briefly, I wish to explore the intuitionist motivation to reject the application of the quantifier exchange rule in question. This is closely related to Dummett?s justificationism, and I will word my explanation accordingly.? If one knows that a predicated characteristic is universally true, this assumes that one either has or may produce a means of obtaining justification of this characteristic.? However, justification of a universal predicate involves justifying a characteristic universally; for the intuitionist, it does not follow that this process of justification, whatever it may be, can automatically provide justification of a particular instance of this predicated characteristic. Thus, contrary to the classical logician, the intuitionist rejects that there is automatically any existentially true instance of a universally true predicate.? On its face, this seems consistent with the intuitionist position; this is precisely the assertion I seek to challenge.

A Discussion of Sets

??????????? As stated at the outset, I wish to argue from intuitionistically acceptable premises.? For this reason, I will spend some time discussing sets, laying out a skeletal definition to be adhered to for the duration of this paper.? My aim is to stay within the scope of constructivist guidelines; however, this is not a clear task.? There are different schools of constructivist thought with varying takes on constructivist mathematics. ?A basic agreement amongst constructivists, and the claim on which I will base the majority of my discussion and my definition of sets, is a similar claim to that made by the justificationists: for an existential claim to be justified, there must be a means of constructing a specific instance of it. *[16]*? I take this to be a central principle of all systems, whether mathematical or logical, under discussion in this paper, and I will outline my set theoretical assumptions in brief accordingly.

(1) Set A, defined by formal axioms, includes members and is determined by its members.

(1) is a basic tenet of set theory,*[17]* and I see no reason why constructivists would object.? It bases the definition of sets on criteria that easily allow for the construction of proofs of claims regarding the set as a whole or its specific members in particular. Similar language to (1) is used by various constructivist schools.? For Russian Constructivists, ?Sets are identified with arithmetic formulas with one free variable.?*[18]*? And for New Constructivists, ?A set S is said to be ?constructed? once recipes are given for constructing S-elements and conditions are specified under which equalities between these elements can be proved.?*[19]*? I take (1) to be consistent with constructivist set theory and would like to add:

(2) Members of A are (or can be) known to be members of A if and only if they can be known to conform to the axiomatic criteria defining A.

(2) is little more than a definition of ?decidable sets.?*[20]*? I take (2) to be a continuation of the New Constructivist position cited above.? I also take (2) to be a rephrasing of the justificationist determination of truth in set-theoretical terms: Z can be justified as a member of A just in case it is possible to produce a means of constructing a proof of Z?s conformity with the axiomatic criteria defining A-membership. As intuitionistically acceptable premises, I accept (1) and (2).

??????????? From these premises and the justificationist requirements for truth, it follows that:

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??????????? (3) The members of A and their characteristics are at least possibly known.

According to (1), A cannot be known without first establishing an axiomatic definition of A.? According to (2), members of A cannot be known to be members of A unless it can be at least possibly shown that they conform to the definition of A-membership. In order to show that Z is a member of A, it must be possible to show that Z adheres to the criteria defining A-membership.? This requires the possibility of knowing that Z possesses the characteristics required for A-membership as well as that no characteristics of Z violate A-membership. It would follow that for Z to be known to be a member of A, the characteristics of Z must be at least possibly known.*[21]*? Having laid the groundwork, so to speak, I will now turn to the central issue in this paper: whether the intuitionist rejection of the quantifier exchange rule ~(x)Px ⊦ (∃x)~Px is justified.

A Warehouse Full of Boxes

??????????? I begin by presenting an example and will proceed with an argument by this example.

??????????? Imagine that Dave is a security guard for a very wealthy intuitionist.? His post is at the entrance to a concrete warehouse consisting of a single, empty room.? There is nothing inside this warehouse when Dave arrives for his first day on the job, not so much as a light, shelf or object.? There is one entrance, the entrance in front of which Dave is posted.? As soon as he arrives, he receives his instructions. ?He is to guard this warehouse and all it contains.? The contents will arrive shortly: multiple boxes of highly flammable extremely precious goods.? Dave will verify that each box entering the warehouse is exactly the same, made of the same materials, containing only the same amount of the specified contents, the expensive, flammable substance, and fire-free.? He is then to number each box and bring it into the warehouse himself.? No one else is ever to gain entrance to this warehouse for any reason.? After the boxes have been transported into the warehouse, Dave is to seal the door and stand guard. He is not to open the door under any circumstances.? He will have a walkie-talkie with which to communicate an hourly update to his employer as to the status of the contents in the warehouse.? His update shall communicate whether or not the boxes in the warehouse are fire-free.? If a box is not fire-free, Dave must immediately communicate which number box has caught fire.? His reports should be detailed, exact and completely truthful.

??????????? His first hour on the job goes by, and Dave calls in his report to his employer.? All of the boxes are fire-free.? His employer is happy to hear it.? A second hour goes by, and his report is the same.? However, during his third hour on the job, catastrophe strikes: Dave sees smoke seeping from the crack beneath the door to the warehouse.? He immediately radios his employer to let him know that the warehouse has caught fire. His employer replies that this is impossible;

the warehouse is made of inflammable materials.? Dave repeats that he sees smoke coming from inside the warehouse, and that he is sure that there is a box in there that is not fire-free. His employer immediately scolds him for entering the warehouse.? Dave answers that he did not enter the warehouse; he did not even open the door. His employer asks him what number box is on fire.? Dave tells him he does not know.? His employer then asks: How does Dave know that there is a box in that warehouse that is not fire-free?

??????????? This is the question I would like to consider: How *does* Dave know that there is a box inside that warehouse that is not fire-free?

It should be clear where I am heading with this example.? The warehouse is a set, W, of boxes, B_1, B_2 . . . B_n.? W can be properly called a set because it satisfies (1).*[22]*

(1a) Set W, defined by formal axioms, includes members and is determined by its members.

A box does not enter the warehouse unless it is directly verified by Dave to conform to the necessary criteria defining W-membership.? B_1, B_2 . . . B_n can properly be called members of W because they satisfy (2).

(2a) Members of W are (or can be) known to be members of W if and only if they can be known to conform to the axiomatic criteria defining W.

All of the boxes in the warehouse, all of the members of W, satisfy (3) as well.

(3a) The members of W and their characteristics are at least possibly known.

W and its members are in complete conformity with the assumptions of this paper.

Dave?s plight may be put in symbolic terms:

??????????? (4) Hour One: (x) Fx (where F stands for ?fire-free? and x is a member of set W)

??????????? (5) Hour Two: (x) Fx

??????????? (6) Hour Three: ~(x) Fx

??????????? (7) Dave?s report at hour three: (∃x) ~Fx

??????????? As he is an intuitionist, Dave?s employer does not accept (7).? How can Dave be sure that there is a box that is not fire-free simply because the whole set of boxes is not fire-free?? Did Dave see the box that is on fire?? No, and he cannot report which number box is on fire.? So how can he justify his claim that there is a box in that warehouse that is not fire-free?? Without failing in his duties, Dave cannot verify with his own eyes which box is on fire. That would require him to enter the warehouse, and he is forbidden to do so.? He cannot let anyone else verify it for him, because he has been instructed not to let anyone else enter the warehouse.? Dave cannot even open the door and peer inside to find the burning box because he has been instructed to leave

the door sealed.? Dave?s other senses are similarly restricted or useless. All of Dave?s possible means of direct verification, while conceivable, would require him to fail in his duties to his employer.? Yet, by being unable to truthfully, as his employer, the intuitionist, defines truth, report that there is a box of a particular number that is not fire-free, Dave is also failing in his duties.? He is trapped in a paradox.

??????????? Or is he?? Is there no way to convince his employer that there is in fact a box that is not fire-free in the warehouse?? His employer is a reasonable man; perhaps he will be convinced by a sound argument concluding just that.

Justifying the Existence of a Burning Box

Other than in the case of single-member sets, the set as a whole is different than the members of the set, and Dave has directly verified that the warehouse contains more than one box, so we know:

(8) W has more than one member.?

It would follow that inferring a characteristic of W as a whole would involve a different process than inferring a characteristic of a single, or even every single member of W; this is what is involved in the fallacy of composition.[23]? But this fallacy relies on improper inference.? If the not fire-freeness of W were simply a characteristic of W (as a whole set), then Dave?s reasoning would be fallacious.? However, we know:

(9) The building itself is not flammable.?

Furthermore, since W has existed for some time while maintaining its fire-freeness, we know:

(10) The not fire-freeness of W is not a result of the grouping of the members of W. (See footnote 3)

Dave directly justifies that the smoke is coming from inside W.? From (9), (10) and our knowledge that W contains only flammable boxes, the predicated not fire-freeness of W must rely on the not fire-freeness of at least one individual box inside W.? Thus:

(11) The not fire-freeness of W is due to the not fire-freeness of some B_n.

We still do not have what we need; we must justify our claim that there exists a B_n that is not fire-free. ?Dave has directly justified that W exists.? From (1a), the existence of some B_n is also justified; a set W cannot exist without members.? From (2a), we know that B_n is a flammable box.? Knowability implies provability (see footnote 18), so, from (3a), we can demonstrate or communicate a proof of the characteristics of B_n.? Truth is justified by both the possibility of proof and direct justification.? Dave has directly justified the not fire-freeness of W. From (9) and (10), we have (11).? Hence:

(12) Justification of the not fire-freeness of an existent W implies justification of the not fire-freeness of an existent B_n.

Dave has a sound argument. His observation ~(x) Fx at hour three justifies his report (∃x) ~Fx.

Consideration of Some Problems

??????????? There remains an issue to be taken up by intuitionists in response to this argument: What is the number of the box that is not fire-free? There are two facets to this objection, and I will attend to them respectively.? Firstly, even if granted that Dave successfully justified that there exists *some* particular box that is not fire-free, he is still unable to prove the particular instance of this box. Dave cannot report a number to his employer.? I answer this with a reference to (i): justification may in principle be available while not practically possible.? This is the position in which Dave has been left.? He can tell his employer exactly how it would be possible to verify the existence of a box that is not fire-free; he could even outline in detail a method for doing so and effectively communicate this to his employer. However, due to the restrictions placed on him by his employer, this means of justification is not practical.

??????????? The second part of this objection goes much deeper.? I mentioned in the beginning of this paper that I did not want to engage in the realist--anti-realist debate; nevertheless, it is unavoidable.? The intuitionist will always have recourse to a strict justificationism: he may require direct justification.? Dummett does not stick to his guns in so strict a sense, but he may if he chose to. His justificationist position is in response to a larger question; for Dummett, ?it is very obscure what a realist?s conception of truth is.?*[24]*? Intuitionist opinions regarding such axioms of classical logic as the principle of bivalence and the quantifier exchange rule ~(x)Px ⊢ (∃x)~Px are based on a justificationist theory of truth.? They reject the classical axioms because they see no definite statement of truth with which to justify them.? I do not wish to answer to Dummett?s accusation in this paper; I merely want to point out that, as Dummett himself suggests in (i) and (ii), it would be severely limiting if one were to hold unwaveringly to a strict justificationist position.? Past truths would often be

unjustifiable; some truths would be unjustified even by the time their justifications were communicated.? Take this example: Imagine my wife and I are enjoying a picnic next to a river.? I see a runner crossing a bridge downriver from us and wish to point this out to my wife. By the time I get her attention and point her gaze in the direction of the bridge, the runner has already crossed it. Even though she sees her running away from the bridge, and even though she has my testimony to the truth of her having crossed the bridge, according to a strict reading of the justificationist position, my wife is still unjustified in accepting as true that the runner crossed the bridge because she has no direct justification of her having done so.

??????????? In the same article, Dummett offers (i) and (ii) in response to criticism from Christopher Peacocke and acknowledges that his justificationism, without such allowances, ?result[s] in a doctrine very hard indeed to believe,? in fact, ?repugnant,?*[25]* but states that he would rather accept these radical anti-realist conclusions about the past than realism.? I agree with Dummett that rejection of these claims leads to ?repugnant? results.? Strict justificationism proves severely limiting for Dave?s employer.? If he were to hold such a conservative justificationist position, rejecting (i) and (ii), then Dave?s employer would be unjustified in accepting any of Dave?s reports as true.? In fact, he would be wasting his money in hiring a guard to stand watch since he could only accept the truth of directly justified claims.? To be certain of the truth of any or all of his boxes? fire-freeness, he would have to justify the claim directly for himself. On these grounds, I assert that, unless he is to accept the justificationist position expressed in (i) and (ii), Dave?s employer?s requests are absurd.? Dave is unable to fulfill his duties because they are impossible to fulfill.? However, if his employer does subscribe to (i) and (ii), then he should accept (7).? Dave?s argument is sound, and this indirect justification should be guarantee enough for his employer that a direct justification exists.? My suggestion is that intuitionists are left in a difficult position: They either reject (i) and (ii) and accept a repugnant anti-realism, or they accept (i) and (ii) and affirm classical quantifier exchange.

Conclusion

??????????? My aim has been to argue that the intuitionist system does in fact justify the quantifier exchange rule ~(x)Px ⊢ (∃x)~Px.? The example of Dave and the warehouse full of boxes shows that as long as the universal predicate is justifiably true and the predicate is not based on an effect of the grouping of the members of the set, then there is a sound argument from intuitionistically acceptable premises for the existence of a particular member within that set of which the predicated characteristic is true.? Unless the intuitionist wishes to require a strict definition of justification, one that would severely limit truth, especially in light of spatio-temporal complications, he should accept indirect proof of this member?s existence based on sound argument. *[26]*

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Further Reading

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[1] Hutchins, Debby. ?The Intuitionist Challenge.? *Fitch?s Proof and the Prospects for Anti-realism*, p. 81.

[2] Null, infinite and imaginary or otherwise non-real sets will not be considered in these pages.? From now on, ?set? will be taken to mean a finite, existent, membered set.

[3] My aim is twofold: to avoid the fallacies of composition and division and to restrict my inquiry to predicated characteristics relevant to my argument.? Characteristics resulting from the grouping of the members of a set will not be considered. An example of such a characteristic is as follows: If player A is the goalie on the best soccer team in the country, team B, this does not necessarily make him the best goalie in the country.? The predicated bestness of team B is a characteristic of the group as a whole, not of its individual members.? Player A may in fact be the worst goalie in the country but team B?s defense so strong that he receives few shots on goal (or there may be other reasons for the team?s success even with a goalie who is not up to snuff).? I differentiate this from characteristics of a set depending on individual member characteristics: If team B has the fastest front line in the country, this is because the two forwards on the team, C and D, as individuals, are faster than all other forwards in the country.? C and D are part of team B, without them as members of the team, team B would no longer have the fastest front line in the country. As opposed to the former example, this characteristic of team B, their predicated fastest-front-lineness, is based on the individual fastness of C and D.? Team B could retain its bestness without player A as goalie; however, it could not retain its fastest-front-lineness without C and D as forwards.

[4] Haack, Susan. *Deviant Logic*. London: Cambridge University Press, 1974, pp. 91-2.

[5] Hutchins, p. 77.

[6] Dummett, Michael. ?The Philosphical Basis of Intuitionism.? *Truth and Other Enigmas*. Cambridge: Harvard University Press, 1978, p. 227.

[7] Dummett, Michael. ?The Justificationist?s Response to a Realist.? *Mind*, vol. 114, 455, July 2005, p. 673.

[8] Ibid., 683.

[9] Ibid., 673.

[10] Ibid., p. 687.

[11] Ibid., p. 688.

[12] Ibid., p.673.

[13] Ibid., p. 674.

[14] Ibid., 679.

[15] Dummett (1978).

[16] McCarty, David Charles (1998). Constructivism in mathematics. In E. Craig (Ed.), *Routledge Encyclopedia of Philosophy*. London: Routledge. Retrieved December 02, 2005, from http://www.rep.routledge.com/article/Y063

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[18] McCarty (1998)

[19] Ibid.

[20] Hunter, Geoffrey. *Metalogic: An Introduction to the Metatheory of Standard First Order Logic*. Berkeley: University of California Press, 1973, p.16.? Although, Hunter denies knowability in this context.

[21] In Dummett (1978), he discusses the requirement that knowledge of truth entails demonstration of use or communicability of this knowledge; this is what leads to one?s ability to provide a verifiable proof of a statement.? In this paper, I take knowability to imply communicability because this is Dummett?s stance, as I read him.? Thus, discussions of knowability imply the ability to communicate this knowledge. This is why I do not specify that characteristics of Z be both known and communicable; the former implies the latter, at least within the scope of the assumptions of this paper.

[22] W also satisfies the definition of sets in this paper: real, finite and membered.

[23] Detlefsen, Michael, David Charles Mccarty and John B. Bacon (1998). Logical and mathematical terms, glossary of: 'Composition, fallacy of'. In E. Craig (Ed.), *Routledge Encyclopedia of Philosophy*. London: Routledge. Retrieved December 01, 2005, from http://www.rep.routledge.com/article/GLOSSITEM126

[24] Dummett (2005), p. 671.

[25] Ibid., p. 672.

[26] It was brought to my attention by my respondent, Alex Cox, that the acceptability by intuitionist standards of quantifier exchange over decidable sets has elsewhere already been proven.? I thank him for this insight; nevertheless, I would still like to offer this paper as an independent argument in support of this claim.